Lesson 10 - Partial Fractions, part II.

MA 16020 LESSONS 9+10: PARTIAL FRACTIONS

METHOD OF DECOMPOSING INTO PARTIAL FRACTIONS

Given a rational function $\frac{N(x)}{D(x)}$

1. Factor the denominator as much as possible.

2. Write the fraction into decomposition form.

"terms" = terms in the factored denominator

Last class

a) Distinct linear terms like $x-a$ decompose to

$$\frac{A}{x-a}$$

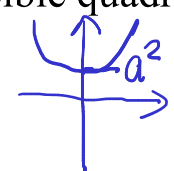
b) Repeated linear terms like $(x-a)^3$ decompose to

$$\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{(x-a)^3}$$

Today

c) Distinct irreducible quadratic terms like x^2+a^2 decompose to

$y=x^2+a^2$



$$\frac{Ax+B}{x^2+a^2}$$

irreducible quadratic
- ax^2+bx+c where $b^2-4ac<0$
- no x-intercepts
- no real solutions to ax^2+bx+c

d) Repeated irreducible quadratic terms like $(x^2+a^2)^2$ decompose to

$$\frac{Ax+B}{x^2+a^2}+\frac{Cx+D}{(x^2+a^2)^2}$$

3. Combine your decomposition from (2) as 1 fraction.

4. Set the original numerator, $N(x)$, equal to the numerator from (3).

5. Equate the coefficients of the terms, to yield a system of equations. Then solve the constants.
   i.e. Find $A, B, C, \ldots$

6. Plug the values found in (5) in (2).

Announcement

Quiz 3 on Wed 9/20

Blue or Black ink

Partial Fractions

$\boxed{\text{EX}}$ $x^2+3x-28$ is reducible

$$b^2-4ac = 3^2-4(1)(-28) > 0$$

$$x^2+3x-28 = (x+7)(x-4)$$

$\boxed{\text{EX}}$

$$x^6-1 = (x^3-1)(x^3+1)$$
$$= (x-1)(x^2+x+1)(x+1)(x^2-x+1)$$
$$= \underbrace{(x+1)(x-1)}_{\text{linear}}\underbrace{(x^2+x+1)(x^2-x+1)}_{\text{irreducible quadratics}}$$

$\boxed{\text{EX}}$ x^2-16 - reducible $\quad x^2-16=(x+4)(x-4)$

x^2+16 - irreducible

$\boxed{\text{EX}}$ Find a PFD (partial fractions decomposition)

Do NOT find the numerical values for the coefficients.

$$\frac{5x+7}{(x+1)^3(x^2+x+1)^2} = \frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{(x+1)^3}+\frac{Dx+E}{x^2+x+1}+\frac{Fx+G}{(x^2+x+1)^2}$$

$b^2-4ac = 1^2-4(1)(1) = -3 < 0$

irred.

$\boxed{Ex}$ $\displaystyle\int \frac{5x^2+12x+10}{x^3+4x^2+5x+20}\,dx$

$$\underbrace{x^3+4x^2}+\underbrace{5x+20}$$
$$x^2(x+4)+5(x+4) = (x^2+5)(x+4)$$

$$\frac{5x^2+12x+10}{(x^2+5)(x+4)} = \frac{Ax+B}{x^2+5} + \frac{C}{x+4}$$
$$= \frac{(Ax+B)(x+4)+C(x^2+5)}{(x^2+5)(x+4)}$$

$$\frac{5x^2+12x+10}{(x^2+5)(x+4)} = \frac{Ax^2+4Ax+Bx+4B+Cx^2+5C}{(x^2+5)(x+4)}$$

$$5x^2+12x+10 = (A+C)x^2+(4A+B)x+(4B+5C)$$

$A + C = 5$ (1)
$4A + B = 12$ (2)
$4B + 5C = 10$ (3)
$B - 4C = -8$ (4)

1st -4(1) + (2)
$-4A - 4C = -20$
$4A + B = 12$
$B - 4C = -8$ (4)

2nd (3) -4(4)
$4B + 5C = 10$
$-4B + 16C = 32$
$21C = 42$
$\boxed{C = 2}$

$C = 2 \rightarrow$ (1)
$A + 2 = 5$
$\boxed{A = 3}$
$A = 3 \rightarrow$ (2)
$4(3) + B = 12$
$\boxed{B = 0}$

$$\int \frac{5x^2+12x+10}{x^3+4x^2+5x+20}\,dx = \int\left(\frac{3x+0}{x^2+5} + \frac{2}{x+4}\right)dx$$

$$\int \left(\frac{3x+0}{x^2+5} + \frac{2}{x+4} \right) dx = \underbrace{\int \frac{3x}{x^2+5}\, dx}_{} + \int \frac{2}{x+4}\, dx$$

$u = x^2+5$

$du = 2x\,dx$

$\frac{1}{2} du = x\,dx$

$$= \int \frac{3 \cdot \frac{1}{2}}{u}\, du + 2 \ln|x+4| + C$$

$$= \frac{3}{2} \ln|u| + 2 \ln|x+4| + C$$

$$= \frac{3}{2} \ln(|x^2+5|) + 2 \ln(|x+4|) + C$$

You try it

$$\int \frac{5x^2+9}{x^3+3x}\, dx$$

$$\frac{5x^2+9}{x^3+3x} = \frac{5x^2+9}{x(x^2+3)} = \frac{A}{x} + \frac{Bx+C}{x^2+3} = \frac{A(x^2+3)+(Bx+C)x}{x(x^2+3)}$$

$5x^2+9 = A(x^2+3) + (Bx+C)x$

$5x^2+9 = Ax^2 + 3A + Bx^2 + Cx$

$5x^2+9 = (A+B)x^2 + Cx + 3A$

$A+B = 5$ (1) $\qquad 3A = 9 \qquad 3 + B = 5$

$C = 0$ (2) $\qquad \boxed{A = 3} \qquad \boxed{B = 2}$

$3A = 9$ (3) $\qquad \boxed{C = 0}$

$$\int \frac{5x^2+9}{x^3+3x}\, dx = \int \left(\frac{3}{x} + \underbrace{\frac{2x}{x^2+3}}_{u = x^2+3} \right) dx = 3 \ln|x| + \ln|x^2+3| + C$$